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Fixed Point Iterations for Multi-Valued Mapping in Uniformly Smooth Banach Space

Samir Dashputre
Associate Professor,
Deptt. of Applied Mathematics,
Shri Shankaracharya Groups of Institutions,
Junwani, Bhilai, (C.G)

Yogesh Kumar Sahu
Assistant Professor,
Deptt. of Applied Mathematics,
Shri Shankaracharya Groups of Institutions,
Junwani, Bhilai, (C.G)

Abstract

Let K be nonempty closed convex bounded subset of a real uniformly smooth Banach space with modulus of smoothness of power type $q > 1$ and $T: K \to P(K)$ a multi–valued quasi-contractive mapping has a fixed point. We proved that the sequences of Mann and Ishikawa iterate converges to a fixed point of T. The result generalizes and extendes former result proved by Sastry and Babu [17].

Keywords: Multi–value mapping, Mann iterates, Ishikawa iterates, fixed points, uniformly smooth Banach space.

Introduction

In 1974, Ciric [8] introduced a new mapping which is known as quasi–contractive mapping. Let K be a nonempty subset of normed linear space X and T is a mapping of K in to itself. Then T is said to quasi–contractive ([8]), if there exists a constant $K \epsilon [0, 1)$ such that

$$\|Tx - Ty\| \leq k \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\} \quad (1.1)$$

for all x, y ϵ K. In [16], Rhoades showed that the contractive definition (1.1) apart from being an obvious generalization of the well known contractive mapping is one of the most general contractive definitions for which Picard iteration give a unique fixed point.

In [15], Rhoades examined the following two iteration processes:

(a) Mann iteration process defind as follows:

For K a convex subset of Banach space X and T be a nonlinear mapping of K into itself, the sequence $\{x_n\}$ in K defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

Where $\{\alpha_n\}$ is sequence in $[0, 1]$ satisfy certain restriction.

(b) Ishikawa iteration process defined as follows:

With K and T as in (a), the sequence $\{x_n\}$ in K defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n T x_n,$$

Where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying certain restriction.

The above iteration processes have been studied by several authors (see e.g., Chidume ([3, 4, 5, 6, 7]), Ishikawa [9], Mann [11], Qihou ([14]), Rhoades ([15]), for approximating solutions of several nonlinear operator equations. Moreover, it is well known that the two schemes may exhibit different behaviours for different classes of nonlinear mappings (see e.g., Rhoades [15]).

The study of fixed point problems for multi–valued mappings was initiated by Kakutani in 1941. Later several authors have worked on this setting. For instance, Nadler jr. [13] has generalized the Banach contraction principle theorem to multi-valued mappings. Subsequently Assad and Kirk [1] have worked with contractive type mapping and Markin [12] pursed the theory to nonexpansive mappings in multi-valued setting and many others in these directions.

Let X is Banach space. A subset of K is called proximinal if for each x ϵ X; there exists an element k ϵ K such that

$$d(x, k) = d(x, k) = \inf\{\|x - y\| : y \epsilon K\},$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We denote the family of all nonempty bounded proximinal subset of K by P (K). Let A, B be two bounded subsets of X. The Hausdroff distance between A and B defined by

$$H(A, B) = \max\{\sup_{a \epsilon A} d(a, B), \sup_{b \epsilon B} d(b, A) : A, B \epsilon P(K)\},$$

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Where $d(a,b) = inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the set B.

Recently, Sastry and Babu [17] extend the convergence results from single valued maps to multi–valued maps by defining the analogs of Mann and Ishikawa iterates for multi-valued maps with a fixed point.

Definition1.1.

(Sastry and Babu [17]). Let K be nonempty convex subset of X, $T: K \to P(K)$ a multi-valued mapping and fix $p \in F(T)$.

(A) The sequence of Mann iterates is defined by $\in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \alpha_n \in [0,1];\ \forall n \geq 0$$

Where $u_n \in Tx_n$ is such that $\|u_n - p\| = d(p, Tx_n)$.

(B) The sequence of Ishikawa iterates is defined by $\in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z'_n, \alpha_n \in [0,1];\ \forall n \geq 0$$

Where $z'_n \in Ty_n$ is such that $\|z'_n - p\| = d(p, Ty_n)$.

$$y_n = (1 - \beta_n)x_n + \beta_n z_n, \beta_n \in [0,1]; \forall n \geq 0$$

Where $z_n \in Tx_n$ is such that $\|z_n - p\| = d(p, Tx_n)$.

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Definition1.2

(Sastry and Babu[17]). A multi–valued $T: K \to P(K)$ is said to be quasi–contractive if for some constant k, $0 \leq k < 1$

$$\|Hx - Hy\| \leq k\, max\{\|x - y\|, d(x,Tx), d(y,Ty), d(y,Tx);\ d(x,Ty)\} \quad (1.2)$$

For all $x, y \in C$. A point x is called fixed point of T if $x \in Tx$.

In [17] Sastry and Babu proved the following theorm:

Theorem 1.3

(Sastry and Babu [17], Theorem 9, p.824). Let K be a closed convex bounded subset of Hilbert space X. Suppose $T: K \to P(K)$ is quasi contractive (1.2) and has a fixed point $p \in F(T)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences such that $0 \leq \alpha_n,\ \beta_n < 1$ for all n and $\beta_n \to 0$ as $n \to \infty$ with $\delta \leq \alpha n \leq 1 - k^2$ for some $\delta > 0$. Then sequence of Ishikawa iterates defined by (B) converges to p.

The main purpose of this paper extend the convergence result of Sastry and Babu [17] from Hilbert space to uniformly smooth Banach space with modulus of smoothness of power type $q > 1$, and also our result extends the convergence result of Chidume and Osilike [7] from single–valued maps to multi-valued maps.

Preliminaries

We begin with the following definition and lemmas:

Let X be a Banach space. The modulus of smoothness of X is the function

$$\rho X : [0,\infty) \to [0,\infty)$$

Defined by

$\rho X(\tau) := \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1; \|y\| \leq \tau\}$.

X is said to be uniformly smooth if:

$\lim_{\tau \to 0} \frac{\rho X(\tau)}{\tau} = 0$:

A uniformly smooth Banach space X is said to have smoothness of power type $q > 1$ if there exists a constant $c > 0$ such that $\rho X(\tau) \leq c\tau^q$.

Let $q > 1$, J_q denotes the generalized duality mapping from X to 2^{X^*} given by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

Where X^* denotes the dual space of X and $\langle . , . \rangle$ denote the generalized duality pairing. In particular J = J_2 is called the *normalized duality mapping* and $J_q = \|x\|^{q-2} J(X)\ if\ x \neq 0$. It is well known that (see e.g., Xu [19]) X is uniformly smooth, if and only if J (and hence J_q) is single valued and uniformly continuous on any bounded subset of X.

In this sequel, we shall need the following lemmas.

Lemma 2.1

([18], Corollary 1, p. 1130) Let X be a uniformly smooth Banach space. Then X has modulus of smoothness of power type $q > 1$ if and only if there exists a constant $c > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q \quad (2.1)$$

for all $x, y \in X$.

For Hilbert space q = 2, c = 1 and equality holds. For $p \geq 2, L_p\ or\ (l_p)$ spaces have modulus of smoothness of power type q = 2 and (2.1) is satisfies with $c = (p - 1)$ (see e.g., Xu [18]).

Lemma 2.2

(Chidume and Osilike [7], p.206) Let X be a uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Then for all $x, y, z \in X$ and $t \in [0,1]$,

$$\|tx + (1-t)y - z\|^q \leq [1 - t(q-1)]\|y - z\|^q + tc\|x - z\|^q - t\left[1 - t^{q-1}c\right]\|x - y\|^q, \quad (2.2)$$

Where $c > 0$ is a constant appearing in (2.1).

Lemma 2.3

(Liu [10], p.118) Let $\{a_n\}\ n \geq 0, \{b_n\}\ n \geq 0, \{c_n\}\ n \geq 0$, and $\{t_n\}\ n \geq 0$, be four sequences of nonnegative numbers satisfying the inequality:

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n\ for\ all\ n \geq 0,$$

where $\{t_n\} n \geq 0 \subset [0,1]$,

$$\sum_{n=0}^{\infty} t_n 65 = \infty, \quad \sum_{n=0}^{\infty} c_n < \infty \quad \text{and } \lim_{n\to\infty} b_n = 0.$$

Then $\lim_{n\to\infty} \alpha_n = 0$.

Main Results

Theorem 3.1.

Let X be a real uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Let K be a nonempty, closed convex and bounded subset of X. Suppose $T : K \to P(K)$ is a multi-valued quasi-contractive mapping and has fixed point p. Let

(i) $c \geq 1, q - 1 \geq ck^q$,

where c is the constant appearing in (2.1).

Let $\{\alpha_n\}$ and $\{\beta_n\}$ real sequences in [0,1] satisfying the condition:

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty} \beta_n = 0$

(iv) $c\alpha_n^{q-1} - ck^q\beta_n(q-1) \leq 1 - ck^q$,

(v) $c\beta^{q-1} \leq \frac{2-ck^q}{2}$.

Then for any $x_0 \in K$, the sequence $\{x_n\}$ defined by (**B**) converges to a fixed point of T.

Proof

Let p be fixed point of T then by using Lemma 2.2 and (B), we have

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$$\|x_{n+1} - p\|^q = \|(1-\alpha_n)x_n + \alpha_n z_n' - p\|^q$$
$$\leq (1-\alpha_n)(q-1)\|x_n - p\|^q + c\alpha_n\|z' - p\|^q$$
$$-\alpha_n(1-\alpha_n^{q-1})\|x_n - z_n'\|^q. \quad (3.2)$$

Again

$$\|z_n' - p\| = d(p, Ty_n)$$
$$\leq \max_{z \in T_p} d(z, Ty_n)$$
$$\leq H(T_p, Ty_n).$$

Therefore,

$$\|z_n' - p\|^q = H^q(Tp, Ty_n)$$
$$\leq k^q max\left\{\|y_n - p\|^q, d^q(y_n, Ty_n), d^q(p, Ty_n)\right\} \quad (3.2)$$

If $d^q(p, Ty_n)$ is maximum, then

$$H^q(Tp, Ty_n) \leq k^q d^q(p, Ty^n$$
$$\leq k^q \max_{z \in T_p} d^q(z, Ty_n)$$
$$\leq k^q H^q(T_p, Ty_n),$$

so that $0 \leq \|z_n' - p\|^q \leq H^q(Tp, Ty_n) = 0$. Hence, from (3.2), we get always

$$\|z_n' - p\| \leq K^q max\left\{\|y_n - p\|^q, d^q(y_n, Ty_n)\right\},$$
$$\leq k^q[\|y_n - p\|^q + d^q(y_n, Ty_n)] \quad (3.3)$$

Using Lemma 2.2, (3.3) and (B), we have

$$\|y_n - p\|^q = \|(1-\beta_n)x_n + \beta_n z_n - p\|^q$$
$$\leq (1-\beta_n(q-1)\|x_n - p\|^q + \beta_n c\|z_n - p\|^q$$
$$-\beta_n\left(1-\beta_n^{q-1}c\right)\|x_n - z_n\|^q \quad (3.4)$$

$$d^q(y_n, Ty_n) \leq \|y_n - z_n'\|^q$$
$$\leq \|(1-\beta_n)x_n + \beta_n z_n - z_n'\|^q$$
$$\leq (1-\beta_n(q-1)\|x_n - z_n'\|^q + \beta_n c\|z_n - z_n'\|^q$$
$$-\beta_n\left(1-\beta_n^{q-1}c\right)\|x_n - z_n\|^q. \quad (3.5)$$

Using (3.3),(3.4) and (3.5), we have

$$\|z_n' - p\|^q \leq k^q(1-\beta_n(q-1)\|x_n - p\|^q$$
$$+ k^q\beta_n c\|z_n - p\|^q$$
$$+ k^q\beta_n c\|z_n - z_n'\|^q\, k^q\left(1-\beta_n(q-1)\right)\|x_n - z_n'\|^q$$
$$-2k^q\beta_n\left(1-\beta_n^{q-1}c\right)\|x_n - z_n\|^q. \quad (3.6)$$

Similar to the inequality (3.3), we get that

$$\|z_n - p\|^q = d^q(p, Tx_n)$$
$$\leq H^q(Tp, Tx_n)$$
$$\leq k^q[\|x_n - p\|^q + d^q(x_n, Tx_n)]. \quad (3.7)$$

From (3.6) and (3.7), we get

$$\|z_n' - p\|^q \leq k^q\|x_n - p\| + k^q\beta_n c\|z_n - z_n'\|^q$$
$$+ k^q(1-\beta_n(q-1))\|x_n - z_n'\|^q$$
$$-k^q\beta_n\left(2-2\beta_n^{q-1}c - ck^q\right)d^q(x_n, Tx_n). \quad (3.8)$$

From (3.1) and (3.8), we have

$$\|x_{n+1} - p\|^q\, (1-\alpha_n(q-1))\|x_n - p\|^q - \alpha_n\left(1-\alpha_n^{q-1}c\right)\|x_n - z_n'\|^q$$
$$+c\alpha_n[k^q\|x_n - p\| + k^q\beta_n c\|z_n - z_n'\|^q$$
$$+k^q\left(1-\beta_n(q-1)\right)\|x_n - z_n'\|^q$$
$$-k^q\beta_n\left(2-2\beta_n^{q-1}c - ck^q\right)d^q(x_n, Tx_n)]$$
$$\leq \left(1-\alpha_n(q-1-ck^q)\right)\|x_n - p\|^q\, c^2\alpha_n\beta_n k^q\|z_n - z^n\|^q$$
$$-ck^q\alpha_n\beta_n\left(2-2\beta_n^{q-1}c - ck^q\right)d^q(x_n, Tx_n)$$
$$-\alpha_n\left[1-\alpha_n^{q-1}ck^q\left(1-\beta_n(q-1)\right)\right]\|x_n - z_n'\|. \quad (3.9)$$

As $\beta_n \to 0$ as $n \to \infty$, there exists a positive integer N_1 such that

$$c\beta_n^{q-1} \leq \frac{2-ck^q}{2}\ \forall\, n \geq N_1$$

so that

$$2-2\beta_n^{q-1}c - ck^q \geq 0\ \forall\, n \geq N_1.$$

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Also we have

$$\alpha_n^{q-1}c - ck^q\beta_n(q-1) \leq 1 - ck^q,$$

Implies that

$$1 - \alpha_n^{q-1}c - ck^q(1-\beta_n(q-1) \geq 0,$$

For all $n \geq N_1$. Consequently, from (3.9), we get that for sufficiently large n

$$\|x_{n+1} - p\|^q \leq \left(1-\alpha_n(q-1-ck^q)\right)\|x_n - p\|^q$$
$$+\alpha_n\beta_n c^2 k^q D, \quad (3.10)$$

Where D is the diameter of C. Now by using lemma 2.3, the sequence $\{x_n\}$ converges to p as $n \to \infty$.

Remark 3.2

For Hilbert spaces $q = 2\ and\ c = 1$, so that if we set $q = 2\ and\ c = 1$ in Theorem 3.1, then condition $(q - 1 - ck^q)$ reduces to $(1 - k^2) < 1$. Moreover, conditions (iii), (iv), and (v) reduce to exactly the same condition of theorem 9 of Sastry and Babu [17].

Theorem 3.3

Let X be a real uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Let k be a closed convex and bounded subset of X. Suppose $T : K \to P(K)$ is a quasi-contractive and has fixed point p. Let $\{\alpha_n\}$ be a real sequence satisfying:

(i) $(i)\ 0 \leq \alpha_n < 1 for\ all\ n \geq 0$

(ii) $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $\sum_{n=0}^{\infty}\alpha_n = \infty$,

Then the sequence $\{x_n\}$ defined by **(A)**, converges to a fixed point of T.

Proof. By using Lemma 2.2 and (A), we get

$$\|x_{n+1} - p\|^q = \|(1-\alpha_n)x_n + \alpha_n u_n - p\|^q$$
$$\leq (1-\alpha_n)(q-1)\|x_n - p\|^q + \alpha_n c\|u_n - p\|^q$$
$$-\alpha_n\left(1-\alpha_n^{q-1}\lambda\right)\|x_n - u_n\|^q \quad (3.11)$$

Similar to inequality, we get

$$\|u_n - p\|^q = d^q(p, Tx_n)$$
$$\leq H^q(Tp, Tx_n)$$
$$\leq k^q\|x_n - p\|^q + d^q(x_n, Tx_n). \quad (3.12)$$

Then From (3.11) and (3.12), we get

$$\|x_{n+1} - p\|^q(1-\alpha_n(q1)\|x_n - p\|^q\alpha_n ck^q\|x_n - p\|^q$$
$$+c\alpha_n k^q d^q(x_n, Tx_n) - \alpha_n\left(1-\alpha_n^{q-1}c\right)\|u_n - x_n\|^q$$
$$\leq \left(1-\alpha_n(q-1-ck^q)\right)\|x_n - p\|^q$$
$$-\alpha_n\left(1-c\,\alpha_n^{q-1} - ck^q\right)d^q(x_n, Tx_n). \quad (3.13)$$

Condition (ii) implies for some N_0 sufficient large $N_0 \in N, \left(1-\alpha_n(q-1-ck^q)\right) < 1\ and$

$\left(1-\alpha_n(q-1-ck^q)\right) \geq 0$ so that from (3.13), we have

$\|x_{n+1} - p\|^q \leq \left(1 - \alpha_n\ (q-1-ck^q)\right)\|x_n - p\|^q$. (3.14)

Iteration of inequality (3.14) from $j = N_0\ to\ N$ yields,

$\|x_{N+1} - p\|^q \leq \boldsymbol{\Pi}_{J=N0}^{N}(1-\alpha_j(q-1ck^q))\|x_0 - p\|^q \to 0$

as n→ ∞ by condition (iii). Hence $x_n \to \infty\ as\ n \to \infty$.

Conclusion

(4.1) We conclude that, by theorems 3.1 and 3.3, either Mann or Ishikawa iterates can be used to approximate the fixed point for multi-valued quasi-contractive mapping in real uniformly smooth Banach space with modulus of smoothness of power type q > 1.

(4.2) Theorems 3.1 and 3.3 extends Theorem [9] of Sastry and Babu [17] from Hilbert space to more general Banach space.

(4.3) Theorems 3.1 and 3.3 extends Theorems 1 and 2 of Chidume and Osilike [7] from single valued maps to multi-valued maps.Fixed Point Iterations for Multi-valued Mapping in Uniformly Smooth Banach Space 9.

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